B. Sc. Part III (Hons) 7^{th} paper

$\boxed{\text{DIFF. EQNs.}}$

Q. Solve $(1-x^2)y_2 - xy_1 - a^2y = 0$, given that $y = e^{a\sin^{-1}x}$ is an integral.

Soln. The given equation

$$y_2 - \frac{x}{1-x^2}y_1 - \frac{a^2}{1-x^2}y = 0$$

It is of the form $y_2 + Py_1 + Qy = R$

$$\therefore P = \frac{-x}{1-x^2}, \ Q = \frac{-a^2}{1-x^2}, \ R = 0.$$

Given that $y = e^{a\sin^{-1}x}$ is an integral.

Let $u = e^{a\sin^{-1}x}$ —— (1)

Let the required general soln be $y = uv$

$\Rightarrow$ v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{-x}{1-x^2} + 2e^{-a\sin^{-1}x}\frac{d}{dx}\left(e^{a\sin^{-1}x}\right)\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{-x}{1-x^2} + \frac{2}{e^{a\sin^{-1}x}} \times e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[\frac{-x}{1-x^2} + \frac{2a}{\sqrt{1-x^2}}\right]\frac{dv}{dx} = 0$$

Put $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\Rightarrow \frac{dz}{dx} + z\left(\frac{-x}{(1-x^2)} + \frac{2a}{\sqrt{1-x^2}}\right) = 0$$

$$\Rightarrow \frac{dz}{dx} = z\left(\frac{x}{1-x^2} - \frac{2a}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dz}{z} = \frac{x}{1-x^2}\,dx - \frac{2a}{\sqrt{1-x^2}}\,dx$$

Integrating, we have

$$\Rightarrow \log z = -\frac{1}{2}\log(1-x^2) - 2a\sin^{-1}x + \log c_1$$

$$\Rightarrow z = \frac{1}{\sqrt{1-x^2}}\, e^{-2a\sin^{-1}x} \cdot c_1 = \frac{dv}{dx} \quad \left[\because z = \frac{dv}{dx}\right]$$

$$\Rightarrow dv = \frac{1}{\sqrt{1-x^2}}\left(e^{-a\sin^{-1}x}\right)^2 \cdot c_1\, dx$$

Put

$$-a\sin^{-1}x = q$$

$$\Rightarrow \frac{-a\,dx}{\sqrt{1-x^2}} = dq$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = \frac{-dq}{a}$$

$$\Rightarrow dv = c_1 (e^{q})^2 \cdot \frac{(-)dq}{a}$$

$$\Rightarrow dv = -\frac{c_1}{a} e^{2q}\, dq$$

integrating we get

$$\Rightarrow v = -\frac{c_1}{2a} e^{2q} + c_2 = -\frac{c_1}{2a} e^{-2a\sin^{-1}x} + c_2$$

$$\Rightarrow v = \frac{-c_1}{2a} e^{-2a\sin^{-1}x} + c_2 \quad \text{——— (2)}$$

Hence, $y = uv$ is the soln where u and v are given by (1) and (2) respectively.